Introduction

Example

Find a quadratic polynomial with roots $7-\sqrt{6}$ and $7+\sqrt{6}$.

Method 1:

$$\begin{aligned}P(x) &= (x-(7-\sqrt{6}))(x-(7+\sqrt{6}))\\ &= x^2-((7-\sqrt{6})+(7+\sqrt{6}))x+(7-\sqrt{6})(7+\sqrt{6})\\ &= x^2-14x+49-6\\ &= x^2-14x+43\end{aligned}$$

Method 2: Notice that the sum of these values is 14 *and their product is* 43*, so a polynomial which will work is* $x^2-14x+43$.

Fact — If $ax^2+bx+c=0$ has roots α and β, then:

$$\begin{aligned}ax^2+bx+c &= a(x-\alpha)(x-\beta)\\ &= a(x^2-(\alpha+\beta)x+\alpha\beta)\\ &= ax^2-a(\alpha+\beta)x+a\alpha\beta\end{aligned}$$

In particular, $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

Tip

We don't get anything for free here!

Suppose we know $\alpha+\beta=5, \alpha\beta=4$*, then to solve this system of equations we can use the fact that* $\beta=\frac{4}{\alpha}$ *to find* $\alpha+\frac{4}{\alpha}=5 \Rightarrow \alpha^2-5\alpha+4=0$*, and we are stuck with a quadratic to solve!*

Example

Suppose α and β are roots of $2x^2 + 4x - 5 = 0$, find

(a) $\alpha^2 + \beta^2$

(b) $\frac{1}{\alpha} + \frac{1}{\beta}$

Note that $\alpha + \beta = -\frac{b}{a} = -2$ *and* $\alpha\beta = \frac{c}{a} = -\frac{5}{2}$, *so*

(a)

$$\begin{aligned}\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (-2)^2 - 2\left(-\frac{5}{2}\right) \\ &= 4 + 5 = 9\end{aligned}$$

(b)

$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} &= \frac{\beta + \alpha}{\alpha\beta} \\ &= \frac{-2}{-\frac{5}{2}} \\ &= \frac{4}{5}\end{aligned}$$

Example

Find a quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ from the previous example.

Method 1: *We've already discovered that* $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{4}{5}$ *we can spot that* $\frac{1}{\alpha} \cdot \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{-\frac{5}{2}} = -\frac{2}{5}$, *so our equation could be:*

$$x^2 - \frac{4}{5}x - \frac{2}{5} = 0$$

Method 2: *Notice that if* $u = \frac{1}{\alpha}, \alpha = \frac{1}{u}$ *or if* $u = \frac{1}{\beta}, \beta = \frac{1}{u}$. *Therefore* $\frac{1}{u}$ *satisfies* $2(\frac{1}{u})^2 + 4(\frac{1}{u}) - 5 = 0 \Rightarrow$ $2 + 4u - 5u^2 = 0$

Cubic Equations

Fact — If $ax^3 + bx^2 + cx + d = 0$ has roots α, β, γ, then

$$\begin{aligned} \alpha + \beta + \gamma &= -\frac{b}{a} \\ \alpha\beta + \beta\gamma + \gamma\alpha &= \frac{c}{a} \\ \alpha\beta\gamma &= -\frac{d}{a} \end{aligned}$$

Tip

These are all **symmetric** functions in α, β, γ, ie if you swap any pair of α, β, γ you still have the same expression

Proof:

$$\begin{aligned} 0 &= ax^3 + bx^2 + cx + d \\ &= a(x-\alpha)(x-\beta)(x-\gamma) \\ &= a(x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma) \end{aligned}$$

Example

Suppose α, β, γ are roots of $x^3 + 4x + 2 = 0$, find

(a) $\alpha^2 + \beta^2 + \gamma^2$

(b) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

Since α, β, γ are roots of $x^3 + 4x + 2 = 0$, $\alpha + \beta + \gamma = -0, \alpha\beta + \beta\gamma + \gamma\alpha = 4, \alpha\beta\gamma = -2$

(a)

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) \\ &= 0^2 - 2 \cdot 4 \\ &= -8 \end{aligned}$$

(b)

$$\begin{aligned} \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} \\ &= \frac{4}{-2} \\ &= -2 \end{aligned}$$

Quartic Equations

Fact — If $ax^4 + bx^3 + cx^2 + dx + e = 0$ has roots $\alpha, \beta, \gamma, \delta$, then

$$\begin{aligned} \sum \alpha &= -\frac{b}{a} \\ \sum \alpha\beta &= \frac{c}{a} \\ \sum \alpha\beta\gamma &= -\frac{d}{a} \\ \alpha\beta\gamma\delta &= \frac{e}{a} \end{aligned}$$

Example

Suppose $\alpha, \beta, \gamma, \delta$ are roots of $x^4 + 7x^2 + 2x + 1 = 0$, find a polynomial with roots: $\alpha - 1, \beta - 1, \gamma - 1, \delta - 1$

Method 1: $(\alpha-1)+(\beta-1)+(\gamma-1)+(\delta-1) = \alpha+\beta+\gamma+\delta-4 = -4$, $(\alpha-1)(\beta-1)+(\alpha-1)(\gamma-1)+\cdots = \sum\alpha\beta-3\sum\alpha+6$, *etc. Very tricky.*

Method 2: If $f(x) = x^4 + 7x^2 + 2x + 1$, *then notice that* $f(\alpha) = 0, f(\beta) = 0, \ldots$. *Now consider* $F(x) = f(x+1)$, *then notice* $F(\alpha - 1) = f((\alpha - 1) + 1) = f(\alpha) = 0$ *etc, so we should try:*

$$\begin{aligned} F(x) &= f(x+1) \\ &= (x+1)^4 + 7(x+1)^2 + 2(x+1) + 1 \\ &= x^4 + 4x^3 + 6x^2 + 4x + 1 + 7(x^2 + 2x + 1) + 2x + 2 + 1 \\ &= x^4 + 4x^3 + 13x^2 + 20x + 4 \end{aligned}$$

Example (PPQ)

The cubic equation

$$z^3 + pz^2 + 6z + q = 0$$

has roots α, β and γ.

(a) Write down the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. *(1 mark)*

(b) Given that p and q are real and that $\alpha^2 + \beta^2 + \gamma^2 = -12$:

  (i) explain why the cubic equation has two non-real roots and one real root; *(2 marks)*

  (ii) find the value of p. *(4 marks)*

(c) One root of the cubic equation is $-1 + 3i$.

  Find:

  (i) the other two roots; *(3 marks)*

  (ii) the value of q. *(2 marks)*

(a) 6

(b) *(i)* *Since the square of real numbers can only be positive, if none of α, β, γ were non-real, then $\alpha^2+\beta^2+\gamma^2 \geq 0$. Therefore at least one root is non-real. However, since non-real roots of polynomials with real coefficients come in complex-conjugate pairs, there must be two non-real roots.*

(ii)

$$\begin{aligned}(\alpha + \beta + \gamma)^2 &= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) \\ &= -12 + 2 \cdot 6 \\ &= 0\end{aligned}$$

Therefore $p = -(\alpha + \beta + \gamma) = 0$.

(c) *(i)* *One of the other roots will be $-1 - 3i$, since it is the complex conjugate. We also have the sum of the roots is zero, so the other root must be 2.*

(ii) $q = -\alpha\beta\gamma = -2(-1 + 3i)(-1 - 3i) = -2(1 + 9) = -20$

Example (PPQ) (a) Show that $(1+i)^3 = 2i - 2$. *(2 marks)*

(b) The cubic equation
$$z^3 - (5+i)z^2 + (9+4i)z + k(1+i) = 0$$
where k is a real constant, has roots α, β and γ.

It is given that $\alpha = 1+i$.

(i) Find the value of k. *(3 marks)*

(ii) Show that $\beta + \gamma = 4$. *(1 mark)*

(iii) Find the values of β and γ. *(5 marks)*

(a)

$$\begin{aligned}(1+i)^3 &= 1 + 3i + 3i^2 + i^3 \\ &= 1 + 3i - 3 - i \\ &= 2i - 2\end{aligned}$$

(b) *(i) Let $f(z) = z^3 - (5+i)z^2 + (9+4i)z + k(1+i)$, then since $f(1+i) = 0$ we must have*

$$\begin{aligned}0 &= (1+i)^3 - (5+i)(1+i)^2 + (9+4i)(1+i) + k(1+i) \\ &= (1+i)((1+i)^2 - (5+i)(1+i) + (9+4i) + k) \\ &= (1+i)(2i - (4+6i) + (9+4i) + k) \\ &= (1+i)(5+k)\end{aligned}$$

Therefore $k = -5$.

(ii) Since $\alpha + \beta + \gamma = 5 + i$ we must have $\beta + \gamma = 5 + i - (1+i) = 4$

(iii) We must also have that $\alpha\beta\gamma = 5(1+i) \Rightarrow \beta\gamma = 5$. This means β, γ are roots of the quadratic $x^2 - 4x + 5 = 0 \Rightarrow (x-2)^2 + 1 = 0$, therefore $\beta, \gamma = 2 \pm i$